DE LA RECHERCHE À L'INDUSTRIE

Inverse problem for seismic tomography

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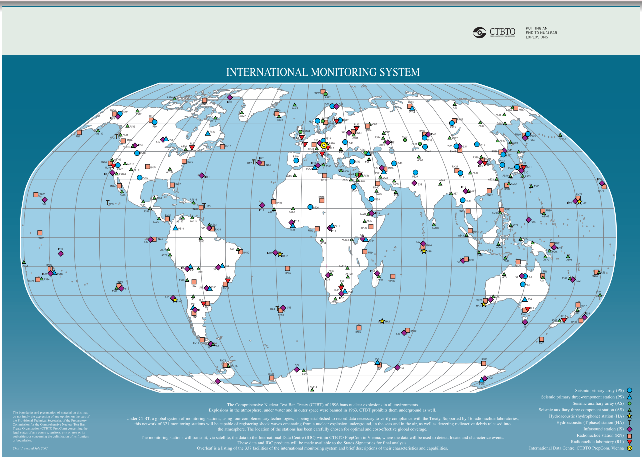
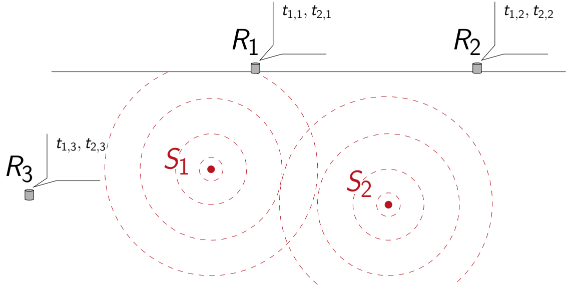
- International treaties (CTBT, NTP)
- Environment monitoring (IMS)
- Detection and analysis of events



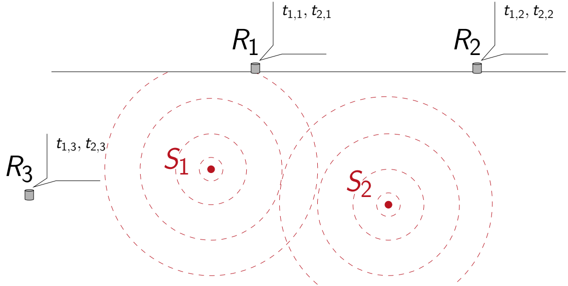


Eikonal solver from Mines ParisTech, Fontainebleau

(source s, receptor r, velocity field v) $\to$ time of arrival $t(s, r, v)$

$$\forall \boldsymbol{x} \in \Omega, \qquad |\nabla t(\boldsymbol{x})|^2 = \frac{1}{v^2(\boldsymbol{x})},$$

- Finite difference operators
- Global fast sweeping method



Eikonal solver from Mines ParisTech, Fontainebleau

(source s, receptor r, $\underbrace{\text{velocity field } v}_{\text{unknown}}$) $\rightarrow$ time of arrival $t(s, r, v)$

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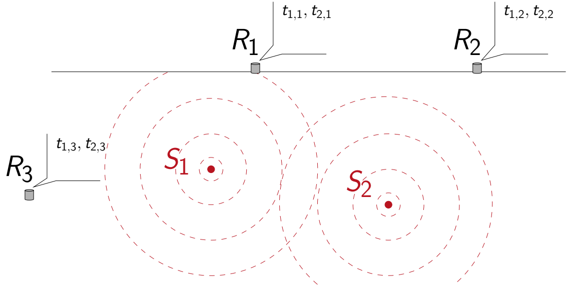


Inverse problem

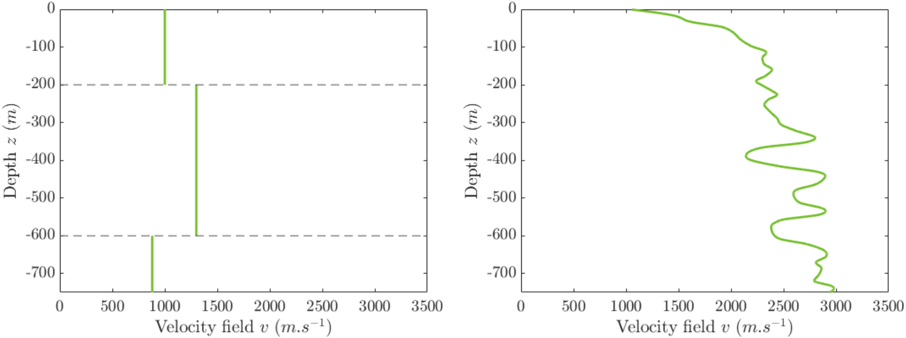
(source s, receptor r, time of arrival $t(s,r,v)$) $\to$ velocity field v

Find v^* such that

$$v^* = \underset{v \in \mathcal{V}}{\text{argmin}} \left\| t_{\text{true}} - t(v,\cdot,\cdot) \right\|^2$$

Questions for seismic tomography

- How to model the velocity field ?
- How to parametrize this velocity model ?
- How to take into account the observations uncertainties ?
- How to propagate the observations uncertainties to the velocity field estimation ?
- How to deal with reasonable computational cost ?



Layered velocity field

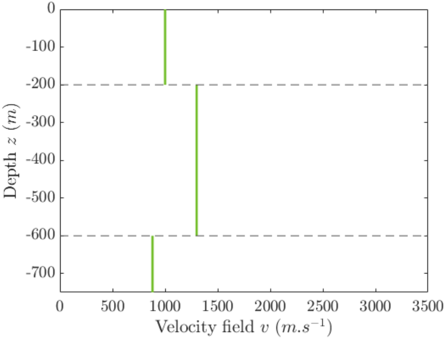
- ✓ Few parameters
- ✗ Strong *a priori*

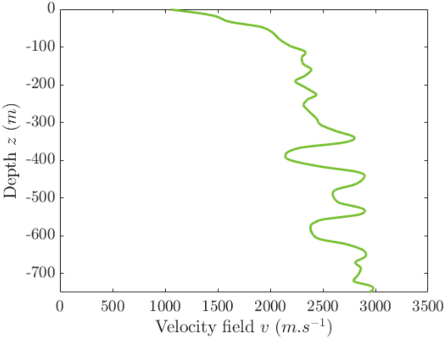
[sochala2021, sochala2021]

Continuous velocity field

- ∼ Large number of (hyper)parameters
- ✓ General shape, weak *a priori*

Modeling the velocity field





Layered velocity field

- ✓ Few parameters
- ✗ Strong *a priori*

[sochala2021, sochala2021]

Continuous velocity field

- ∼ Large number of (hyper)parameters
- ✓ General shape, weak *a priori*

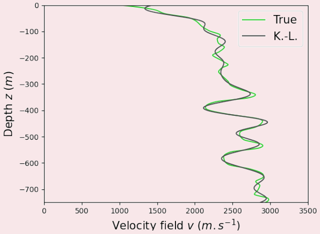
⇝ How to reduce the dimension ?

$u(z) = \log(v(z))$ is seen as a particular realization of a Gaussian process $U \sim \mathcal{N}(\mu, k(\cdot,\cdot))$, where k is the covariance kernel

Karhunen–Loève truncation

$$u(z) = U(z,\omega) \simeq \mu + \sum_{i=1}^{r} \sqrt{\lambda_i} u_i(z) \eta_i(\omega), \ \boldsymbol{\eta} := (\eta_i)_{1 \leqslant i \leqslant r} \sim \mathcal{N}(0, \mathrm{I}_r)$$

- $(u_i, \lambda_i)_{i \in \mathbb{R}_+}$ eigenelements of k:
  $$\langle k(z,\cdot), u_i \rangle_\Omega := \int_\Omega k(z,z') u_i(z') dz' = \lambda_i u_i(z)$$
- Bi-orthonormality: $\forall i,j \in \mathbb{R}_+^*$,
  $\langle u_i, u_j \rangle_\Omega = \delta_{i,j}$,
  ${}_\omega(\eta_i) = 0$ and ${}_\omega(\eta_i \eta_j) = \delta_{i,j}$.



Squared-exponential kernel

$A \in \mathbb{R}_+$ amplitude, $\ell \in \mathbb{R}_+$ length of correlation

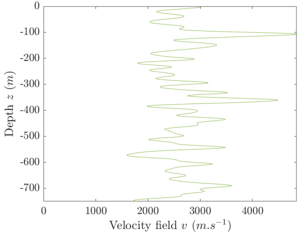
$$k(z, z', q = \{A, \ell\}) = A \exp\left(\frac{-\|z - z'\|^2}{\ell^2}\right)$$

Dependency of hyperparamaters

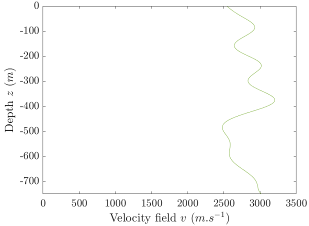
$\rightsquigarrow$ v depends on hyperparameters $q \in \mathbb{H} \subset \mathbb{R}_+ \times \mathbb{R}_+$:

$$\Rightarrow \mathcal{V} = \left\{ v(\mu, \boldsymbol{\eta}, q) = \exp\left(\mu + \sum_{i=1}^{r} \sqrt{\lambda_i(q)} u_i(q) \eta_i\right), \right.$$
$$\left. (\mu, \boldsymbol{\eta}, q) \in \mathbb{R}^r \times \mathbb{R} \times \mathbb{H} \right\}$$

- Choice of q *a priori*
  - ✘ Selection, strong *a priori*, bad *a posteriori* estimation
  - ✓ Easy to implement



(a) $\ell = 10$



(b) $\ell = 70$

Figure: Two fields obtained with same parameters except length of correlation ℓ

Modeling the velocity field: hyperparameters

- Choice of q *a priori*
  - ✘ Selection, strong *a priori*, bad *a posteriori* estimation
  - ✓ Easy to implement
- Change of coordinates [sraj2016, sraj2016]
  - ✓ Keep weak *a priori*
  - ✘ q-dependent eigenvalue problem at each MCMC step
- Change of measure
  - ✓ Keep weak *a priori*
  - ✓ No eigenvalue problem during the MCMC sampling

Current problem

The problem is reduced to find $X^* := (\mu^*, \boldsymbol{\eta}^*, q^*)$ s.t.

$$X^* = \underset{X \in \mathbb{R} \times \mathbb{R}^r \times \mathbb{H}}{\operatorname{argmin}} \| t_{\mathrm{true}} - t(v(X), \cdot, \cdot) \|^2$$

Modeling the velocity field: change of measure

$$U_q = \mu + \sum_{i=1}^{r} \sqrt{\lambda_i(q)} u_i(q) \eta_i, \qquad \boldsymbol{\eta} \sim \mathcal{N}(0, \mathrm{I}_r)$$

Reference basis

$$U_q \simeq U_q = \mu + \sum_{i=1}^{r} \sqrt{\overline{\lambda}_i} \overline{u}_i \xi_i, \qquad \boldsymbol{\xi} := (\xi_i)_{1 \leqslant i \leqslant r} \sim \mathcal{N}(0, \Sigma(q))$$

where $(\overline{u}_i, \overline{\lambda}_i)_{1 \leqslant i \leqslant r}$ are the eigenelements of $\overline{k}$

$$\overline{k}(z, z') = \int_{\mathbb{H}} k(z, z', q) p_{\mathrm{prior},q}(q) dq,$$

Covariance matrix

$\Sigma(q)$ is the projection of $k(\cdot, \cdot, q)$ on the reference basis:

$$\Sigma(q)_{ij} = (\overline{\lambda}_i \overline{\lambda}_j)^{-1/2} \langle \langle k(\cdot, z', q), \overline{u}_j \rangle_\Omega, \overline{u}_i \rangle_\Omega$$

$\Rightarrow$ The problem is reduced to find $Y^* := (\mu^*, \boldsymbol{\xi}^*, q^*)$

Proposition

This approximation minimizes the error of representation in the L^2-sense [sraj2016, sraj2016], $\forall z, z' \in \Omega$,

$$_q \left[_\omega \left(U_q^r(z,\cdot) U_q^r(z',\cdot) \right) - _\omega \left(U_q^r(z,\cdot) U_q^r(z',\cdot) \right) \right] = 0.$$

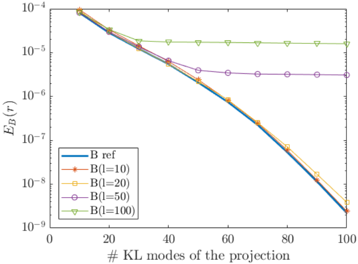


Figure: Mean (over ℓ) normalized error according to r and $\mathcal{B}$.

Averaged error along $q = \{\ell\}$ when projecting on basis $\mathcal{B}$ fields $v_i(\ell)$:

$$E_{\mathcal{B}}(r) = \frac{1}{|\mathbb{H}|} \sum_{\ell \in \mathbb{H}} \frac{\sum_i \| v_i(\ell) - \langle v_i(\ell), \mathcal{B} \rangle \|_{L^2}^2}{\sum_i \| v_i(\ell) \|_{L^2}^2}$$

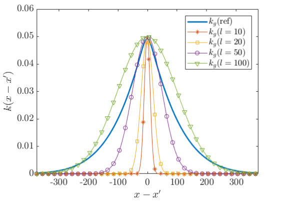
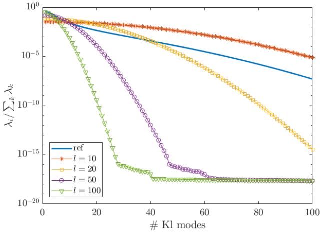
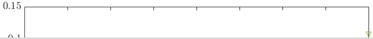


Figure: Covariance kernels



(a) Eigenvalues



Model for the measurement errors

Problem: t_{true} is unknown, obervations are noised:

$$t_{\text{true}} = t_{\text{obs}} + \varepsilon_{\text{obs}}$$

Noise model

Observations are considered i.i.d.

$$\varepsilon_{\text{obs}} \sim \mathcal{N}(0, \alpha^2 \mathrm{I}_{N_{\text{obs}}}),$$

$\alpha \in \mathbb{R}_+^\star$ is the absolute noise level

$\Rightarrow$ The problem becomes to find $Y^* := (\mu^*, \boldsymbol{\xi}^*, q^*, \alpha^*)$ such that

$$Y^* = \underset{Y \in \mathbb{R}^r \times \mathbb{R} \times \mathbb{H} \times \mathbb{R}_+^\star}{\operatorname{argmin}} \frac{\|t_{\text{obs}} - t(Y)\|^2}{\alpha^2}$$

Bayes' rule

$$p_{\text{post}}(Y|t_{\text{obs}}) \propto \mathcal{L}(t_{\text{obs}}|Y)p_{\text{prior}}(Y),$$

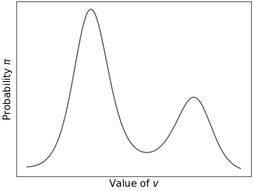
Definition of the likelihood

$$\mathcal{L}(t_{\text{obs}}|Y) = \frac{1}{\sqrt{(2\pi)^N \alpha^{2N}}} \exp\left(-\frac{\|t_{\text{obs}} - t(Y)\|^2}{2\alpha^2}\right)$$

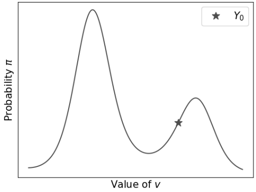
Definition of the prior

$$p_{\text{prior}}(Y) = p_{\text{prior},\alpha}(\alpha)p_{\text{prior},q}(q)p_{\text{prior},\mu}(\mu)p_{\text{prior},\xi}(\xi|q)$$

with $\alpha \sim p_{\text{prior},\alpha}(\alpha) \propto \frac{1}{\alpha}, \quad q \sim \mathcal{U}(\mathbb{H}), \quad \mu \sim \mathcal{U}([\mu_-, \mu_+]), \quad \xi \sim \mathcal{N}(0, \Sigma(q)).$

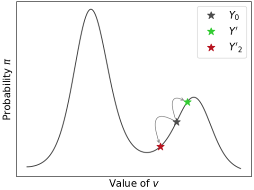


Markov Chain Monte Carlo (MCMC)



Markov Chain Monte Carlo (MCMC)

- Initialization of $Y^{(0)} \to p_{\mathrm{post}}(Y^{(0)})$



Markov Chain Monte Carlo (MCMC)

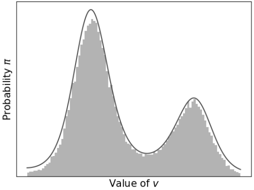
- Initialization of $Y^{(0)} \rightarrow p_{\text{post}}(Y^{(0)})$
- At each step $n+1$:
  - New state proposal $Y_{\text{new}} \rightarrow p_{\text{post}}(Y_{\text{new}})$
  - Metropolis-Hasting criterion: $Y^{(n+1)} = Y_{\text{new}}$

  with probability $\min(1, p_{\text{post}}(Y_{\text{new}})/p_{\text{post}}(Y^{(n)}))$

Use of K-updated covariance matrix proposal [haario2001, haario2001]

$$Y_{\text{new}} \sim \mathcal{N}(Y^{(n)}, C^{(n)})$$

$$\text{with } C^{(n)} = \begin{cases} s_d \mathbb{C}\text{ov}\left(Y^{(0)}, \cdots Y^{(n)}\right) & \text{if } \bmod(n, K) = 0 \\ C^{(n-1)} & \text{else.} \end{cases}$$



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- Posterior distribution

Use of K-updated covariance matrix proposal [haario2001, haario2001]

$$Y_{\mathrm{new}} \sim \mathcal{N}(Y^{(n)}, C^{(n)})$$

$$\text{with } C^{(n)} = \begin{cases} s_d \mathbb{C}\mathrm{ov}\left(Y^{(0)}, \cdots Y^{(n)}\right) & \text{if } \mathrm{mod}(n, K) = 0 \\ C^{(n-1)} & \text{else.} \end{cases}$$

Surrogate of the eikonal solver

Use of eikonal solver into MCMC: one evaluation for each step

Polynomial Chaos expansion

$N(r + |q| + 1)$ eikonal evaluations $\Rightarrow$ Projection
$\Rightarrow$ Surrogate

$$t(v) \equiv t(\mu, \boldsymbol{\xi}, q) \simeq \tilde{t}(\mu, \boldsymbol{\xi}, q) = \sum_{\kappa \in \mathcal{K}} t_\kappa \phi_\kappa(\mu, \boldsymbol{\xi}, q)$$

Change of measure

Parameter independency:

$$p_{\text{post}}(Y|t_{\text{obs}}) \propto \underbrace{\mathcal{L}(t_{\text{obs}}|\mu, \boldsymbol{\xi}, \alpha)}_{\text{depends on } t(\mu,\boldsymbol{\xi})} \quad \underbrace{p_{\text{prior}}(Y)}_{\text{depends on } \Sigma(q)} \quad ,$$

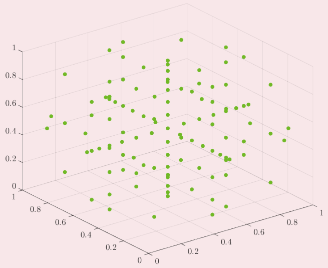
$N(r + |q| + 1) \rightarrow N(r + 1) + N(|q|)$ exact evaluations

Surrogate of the eikonal solver

Parametrized basis

$$\widetilde{t}(\mu, \boldsymbol{\xi}, q)$$
$$N(r + |q| + 1)$$

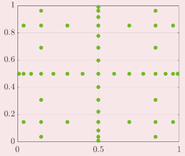
Sparse Grid

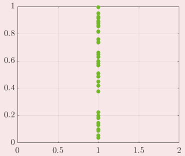


Change of measure

$$\widetilde{t}(\mu, \boldsymbol{\xi}) \text{ and } \widetilde{\Sigma}(q)$$
$$N(r + 1) + N(|q|)$$

Sparse Grid + LHS





Velocity field v

MODEL — KARHUNEN-LOÈVE DECOMPOSITION

$$\downarrow$$

$$(\mu, \eta, q), \ \eta \sim \mathcal{N}(0, 1)$$

PARAMETRIZATION — CHANGE OF MEASURE

$$\downarrow$$

$$(\mu, \xi, q), \ \xi \sim \mathcal{N}\left(0, \tilde{\Sigma}(q)\right)$$

OBSERVATION UNCERTAINTIES — BAYESIAN FRAMEWORK

$$\downarrow$$

$$\mathcal{L}\left(t^{obs} | \mu, \xi, q\right) \propto \|\tilde{t} - t^{obs}\|$$

ESTIMATION UNCERTAINTIES — MARKOV CHAIN MONTE-CARLO

$$\downarrow$$

$$p\left(\mu^\star, \xi^\star, q^\star\right)$$

COST REDUCTION — POLYNOMIAL CHAOS SURROGATES

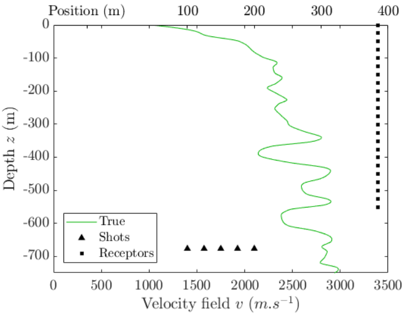


Figure: True field [amoco, amoco] and location of stations

Parameter	Bounds
μ	$[6.9, 8.1]$
ξ	$[-20, 20]^r$
A	$[1.10^{-6}, 1.10^{-1}]$
ℓ	$[10, 150]$
$\log(\alpha)$	$[-10, -3]$
$N_{\mathrm{burn}} = N_{\mathrm{samp}}$	1.10^6
N_{adapt}	5.10^4

Table: Parameters range/value, $r = 20$

- Accuracy: use of RRMSE

$$\mathrm{RRMSE}(Q_{\mathrm{true}}, Q_{\mathrm{approx}}) = \sqrt{\frac{\sum_{i=1}^{N}(Q_{\mathrm{true},i} - Q_{\mathrm{approx},i})^2}{\sum_{i=1}^{N} Q_{\mathrm{true},i}^2}}.$$

- Accuracy: use of RRMSE

$$\mathrm{RRMSE}(Q_{\mathrm{true}}, Q_{\mathrm{approx}}) = \sqrt{\frac{\sum_{i=1}^{N}(Q_{\mathrm{true},i} - Q_{\mathrm{approx},i})^2}{\sum_{i=1}^{N} Q_{\mathrm{true},i}^2}}.$$

- Cost reduction: speed (speed-up factor) and number of exact evaluations

Surrogates analysis

- Accuracy: use of RRMSE

$$\mathrm{RRMSE}(Q_{\mathrm{true}}, Q_{\mathrm{approx}}) = \sqrt{\frac{\sum_{i=1}^{N}(Q_{\mathrm{true},i} - Q_{\mathrm{approx},i})^2}{\sum_{i=1}^{N} Q_{\mathrm{true},i}^2}}.$$

- Cost reduction: speed (speed-up factor) and number of exact evaluations

Eikonal surrogate $\tilde{t}$

Parameters: level 3, $r = 20$
$Q_{\mathrm{true}} = t(v)$
$\mathrm{RRMSE} = 1\%$ on times of arrival
Speed: $30' \rightarrow < 1''$ for 1000 evaluations
Exact evaluations: 15135 instead of 2.10^6

- Accuracy: use of RRMSE

$$\mathrm{RRMSE}(Q_{\mathrm{true}}, Q_{\mathrm{approx}}) = \sqrt{\frac{\sum_{i=1}^{N}(Q_{\mathrm{true},i} - Q_{\mathrm{approx},i})^2}{\sum_{i=1}^{N} Q_{\mathrm{true},i}^2}}.$$

- Cost reduction: speed (speed-up factor) and number of exact evaluations

Eikonal surrogate $\widetilde{t}$

Parameters: level 3, $r = 20$

$Q_{\mathrm{true}} = t(v)$

$\mathrm{RRMSE} = 1\%$ on times of arrival

Speed: $30' \rightarrow < 1''$ for 1000 evaluations

Exact evaluations: 15135 instead of 2.10^6

Covariance matrix surrogate $\widetilde{\Sigma}(q)$

Parameters: level 6 or more, $r = 1$

$Q_{\mathrm{true}} = p_{\mathrm{prior},\xi}(\xi|q)$

$\mathrm{RRMSE} = 1\%$ on ξ prior probability

Speed: $50\times$ speedup

Exact evaluations: 1000 instead of 2.10^6

Results of the inference

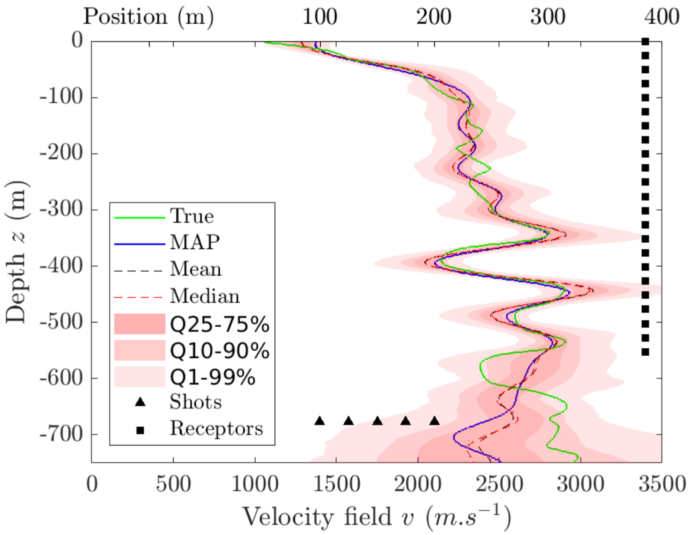
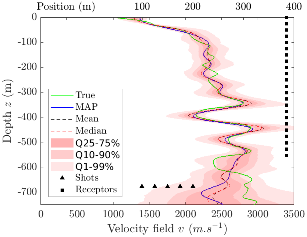
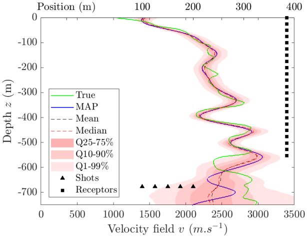


Figure: Posterior distribution compared to true field

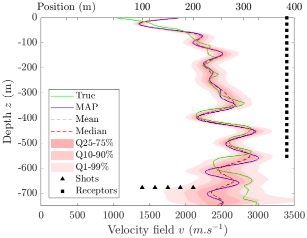
Comparison with *a priori* parametrized field



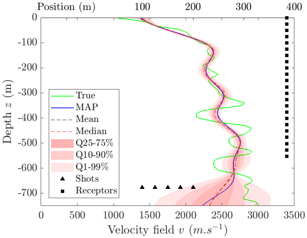
(a) Reference basis



(b) Parametrized basis $\ell = 34$



(c) Parametrized basis $\ell = 10$



(d) Parametrized basis $\ell = 80$

Sources of errors

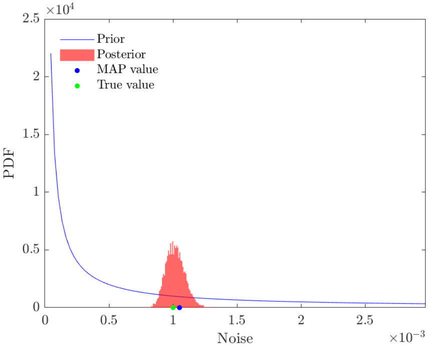
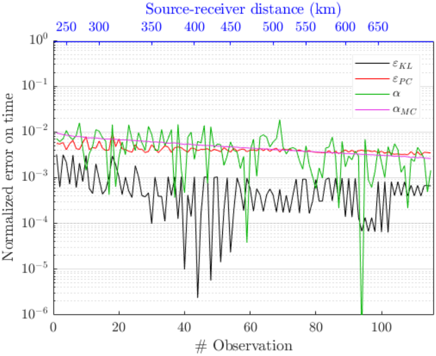


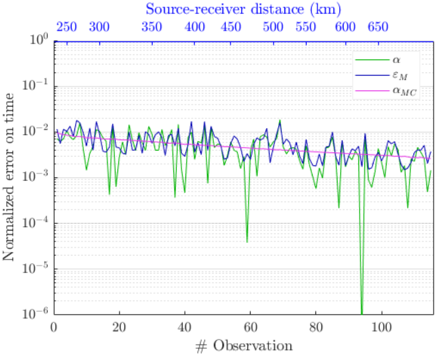
Figure: Posterior distribution for noise level α

$$\varepsilon_{\text{tot}} = \varepsilon_{\text{obs}} + \varepsilon_{\text{eik}} + \underbrace{\varepsilon_{\text{KL}} + \varepsilon_{\text{PC}} + \varepsilon_{\text{MCMC}}}_{:=\varepsilon_M}.$$

Sources of errors



(a) Different sources of errors



(b) Combining model errors

Figure: RRMSE for the different error sources

$$\varepsilon_{\text{tot}} = \varepsilon_{\text{obs}} + \varepsilon_{\text{eik}} + \underbrace{\varepsilon_{\text{KL}} + \varepsilon_{\text{PC}} + \varepsilon_{\text{MCMC}}}_{:=\varepsilon_M} .$$

Other velocity field shapes

Matérn's kernel

$\forall (z, z') \in \Omega^2,$

$$k(z, z', q) = A \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\sqrt{2\nu} \frac{\|z - z'\|^2}{l} \right)^{\nu} K_{\nu} \left(\sqrt{2\nu} \frac{\|z - z'\|^2}{l} \right)$$

- one supplementary parameter ν
- increases possible velocity shape
- does not increase computational cost of the eikonal surrogate

Other velocity field shapes

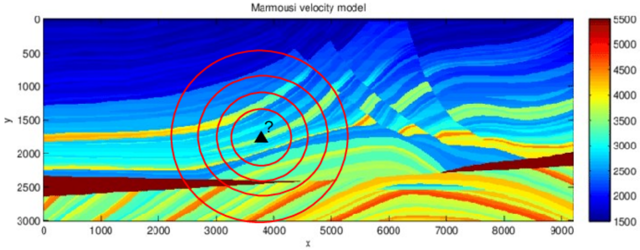
Non constant mean

$\forall z \in \mathbb{R}_+$,

$$v(z) = az^b \exp\left(\sum_{i=1}^{r} \lambda_i^{1/2} u_i(z)\xi_i\right).$$

- $\mu = \log(a) + b\log(z)$
- generalizes the constant mean
- adds a dimension to eikonal surrogate
- geophysically pertinent

- Robustness of the results (toward observations and data)
- Multi-fidelity/adaptive methods
- $2D$ and $3D$ velocity fields
- Uncertainty quantification and model of errors
- Source location



Answers to questions for seismic tomography

- How to model the velocity field ?
  $\rightsquigarrow$ Karhunen–Loeve decomposition (optimal)
- How to parametrize this velocity model ?
  $\rightsquigarrow$ No parametrization thanks to change of measure
- How to take into account the observations uncertainties ?
  $\rightsquigarrow$ Bayesian framework
- How to propagate the observations uncertainties to the velocity field estimation ?
  $\rightsquigarrow$ Markov Chain Monte–Carlo sampling
- How to deal with reasonable computational cost ?
  $\rightsquigarrow$ Surrogate models

Conclusion

- Results infer well the true field
- Uncertainties are in adequation with the geophysical observations
- Surrogates allows the reduction of computational cost
- `Matlab` implementation of the method
- Innovative method in the geophysical field
- Promising for location and two-dimensional fields

References

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Number of sampling grid points

Fejér grid of type II, in dimension r, for a sampling level l, the number of sampling points is equal to

$$n(r,l) = 2^{l+1} \sum_{k=0}^{r-1} \binom{l+k}{l} (-1)^{r-1-k} + (-1)^r.$$

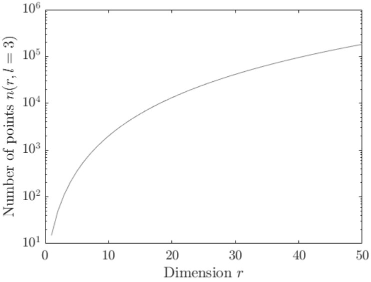
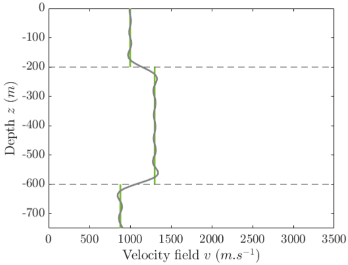
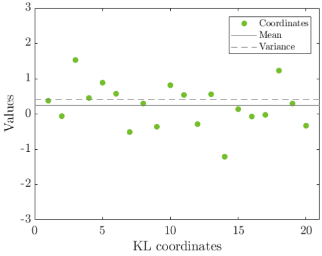


Figure: Size of the Smolyak's sampling grid using Fejér's type II quadrature rule according to the dimension for level $l = 3$.

Discrete field projection



(a) Discrete field projection



(b) Projection's coordinates

Figure: Projection on the reference basis for a discrete field