Hierarchical formulation for Karhunen–Loève parametrization in Bayesian Field inference

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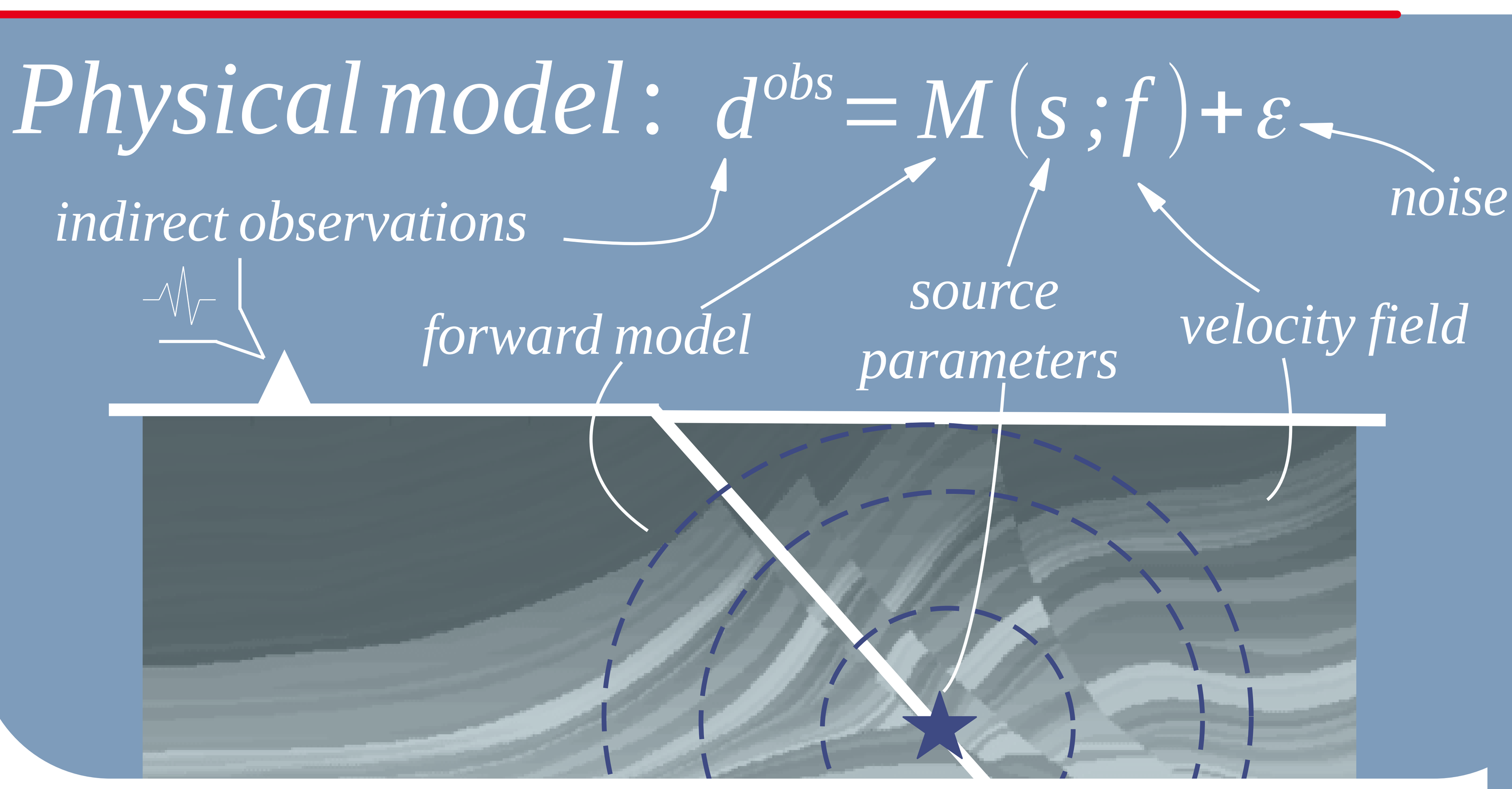
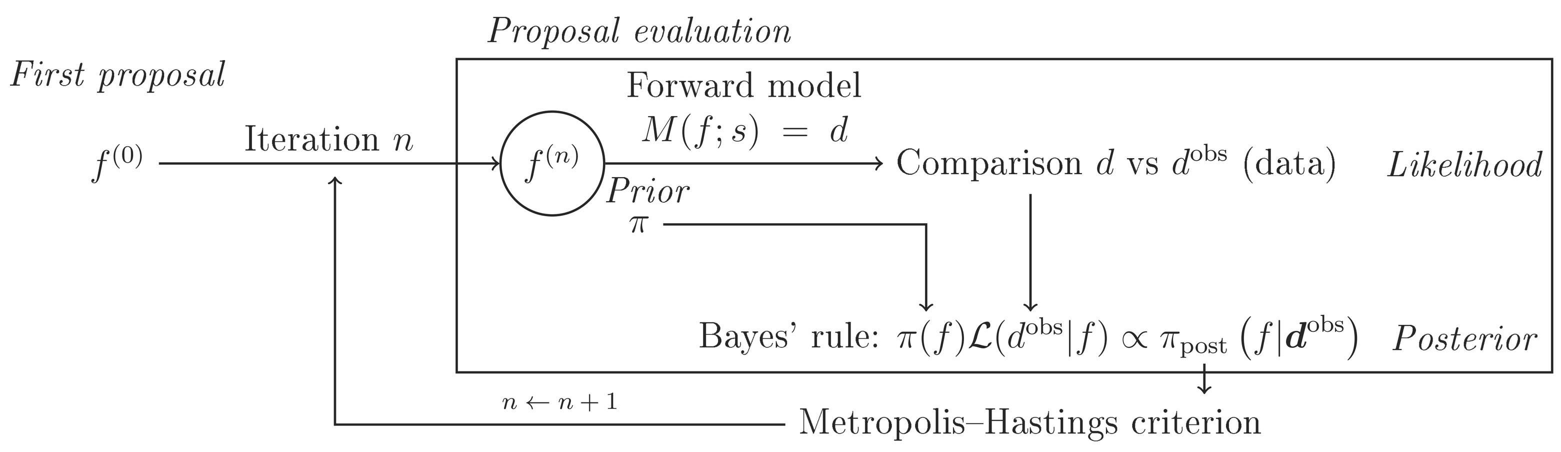
inverse problem | random fields | Karhunen–Loève decomposition | autocovariance hyperparameters | hierarchical Bayes

1 Context

- Earthquake event analysis thanks to **inverse problem** solving
- Problem: the inversion depends on **uncertain model parameters**, *e.g.* the velocity **field**
- Objective: **improvement** of the source parameter **uncertainties**
- Focus: development of a **new parametrization** for the velocity field



2 Mathematical framework



- Goal: to **characterize the field** f thanks to indirect observations d^{obs}, knowing source parameters s
- **Probabilistic** framework: Bayesian inference with Markov Chain Monte Carlo sampling

⇒ **How to express f and its prior in a small dimensional space ?**

3 Karhunen–Loève representation

Assumption: $f \in L^2(\Omega)$ is a particular realisation of a **Gaussian process** $\mathcal{N}(0, k(\boldsymbol{q}))$.
In fact, k depends on hyperparameters $\boldsymbol{q}$ that are difficult to choose *a priori*.

Karhunen–Loève formulation

The field propositions write

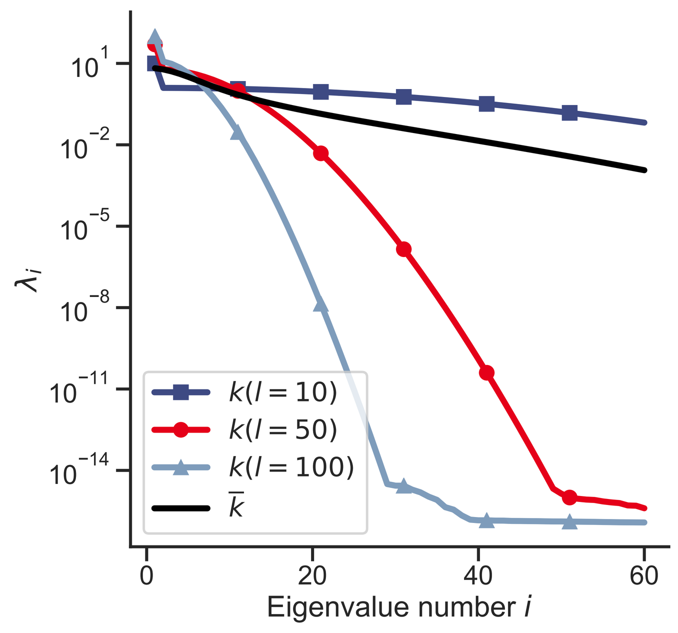
$$f(x, \boldsymbol{\eta}, \boldsymbol{q}) = \sum_{i=1}^{r} \lambda_i(\boldsymbol{q})^{1/2} u_i(x, \boldsymbol{q})\eta_i, \text{ with } \begin{cases} \pi(\boldsymbol{\eta}) \text{ is } \mathcal{N}(0, \mathrm{I}), \\ (\lambda_i(\boldsymbol{q}), u_i(\boldsymbol{q})) \text{ eigenelements of } k(\boldsymbol{q}). \end{cases}$$

If $\boldsymbol{q}$ is not fixed, the posterior quantity becomes

$$\pi_{post}(f(\boldsymbol{\eta}, \boldsymbol{q})|d^{obs}) \propto \mathcal{L}(d^{obs}|f(\boldsymbol{\eta}, \boldsymbol{q}))\pi(\boldsymbol{\eta}, \boldsymbol{q}).$$

How to choose the autocovariance function k ?
For instance, we choose k to be a squared exponential autocovariance function with different correlation lengths l, *i.e.*

$$k(x, y, \boldsymbol{q} = \{l\}) = \exp\left(\frac{-\|x-y\|^2}{2l^2}\right).$$



Eigenvalue decay for different bases

4 Change of measure (CoM) method (in prep.)

⇒ **We present a method to explore the hyperparameters space during the inference, while mitigating the computational cost.**

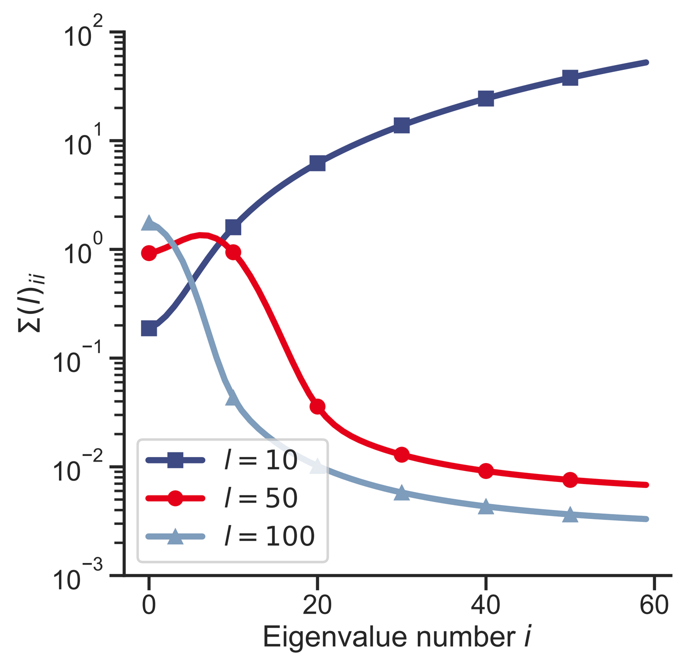
- We use a **reference basis** that does not depend on hyperparameters (Sraj et al., 2016),
- We **transfer the $\boldsymbol{q}$-dependency to the coordinates prior**,

Hierarchical formulation

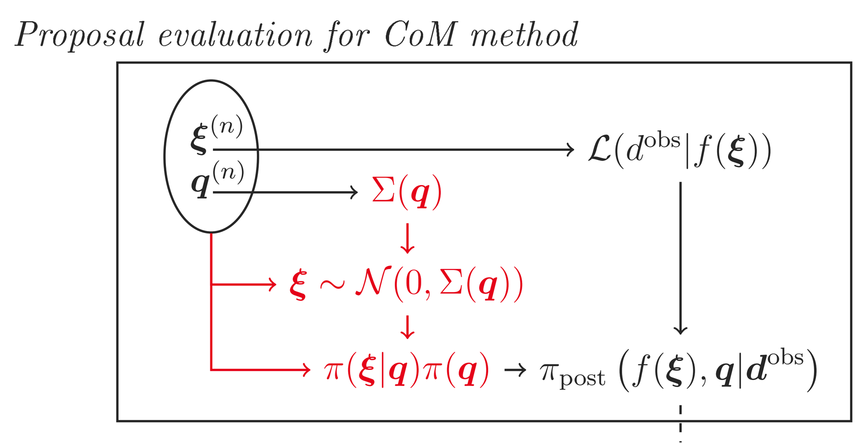
The field propositions write

$$f(x, \boldsymbol{\xi}) = \sum_{i=1}^{r} \overline{\lambda}_i^{1/2} \overline{u}_i(x)\xi_i, \text{ with } \begin{cases} \pi(\boldsymbol{\xi}) \text{ depending on } \boldsymbol{q}, \\ (\overline{\lambda}_i(\boldsymbol{q}), \overline{u}_i(\boldsymbol{q})) \text{ reference eigenelements.} \end{cases}$$

The posterior quantity becomes $\pi_{post}(f(\boldsymbol{\xi}), \boldsymbol{q}|d^{obs}) \propto \mathcal{L}(d^{obs}|f(\boldsymbol{\xi}))\pi(\boldsymbol{\xi}|\boldsymbol{q})\pi(\boldsymbol{q})$,
with $\boldsymbol{\xi} \sim \mathcal{N}(0, \Sigma(\boldsymbol{q}))$, where $\Sigma(\boldsymbol{q})_{ij} = (\overline{\lambda}_i\overline{\lambda}_j)^{-1/2} \langle\langle k(\boldsymbol{q}), \overline{u}_i\rangle_\Omega, \overline{u}_j\rangle_\Omega$,
i.e. $\Sigma(\boldsymbol{q})$ is the double projection of the $\boldsymbol{q}$-dependent kernel on the reference basis.



ξ prior variance.



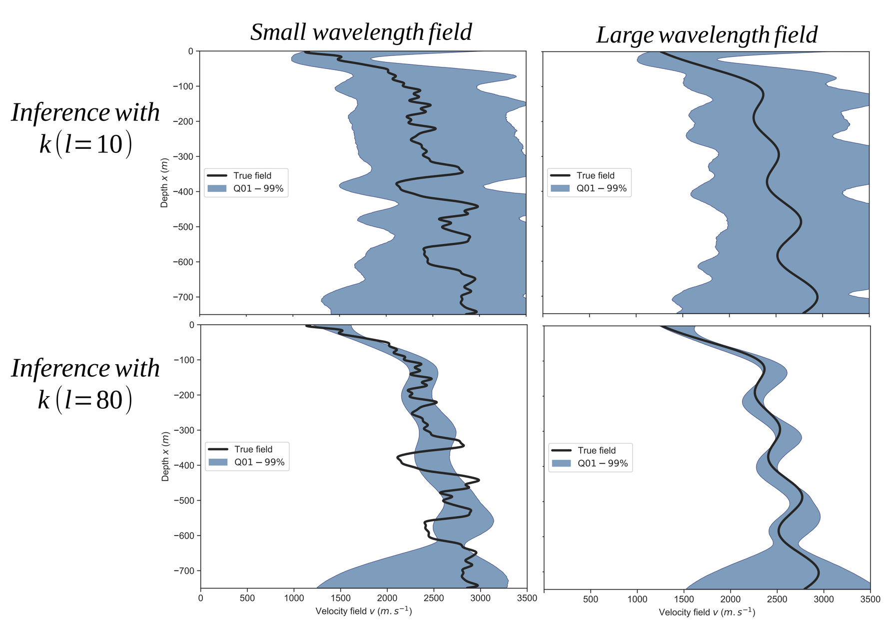
Online workflow for the CoM.

5 Application

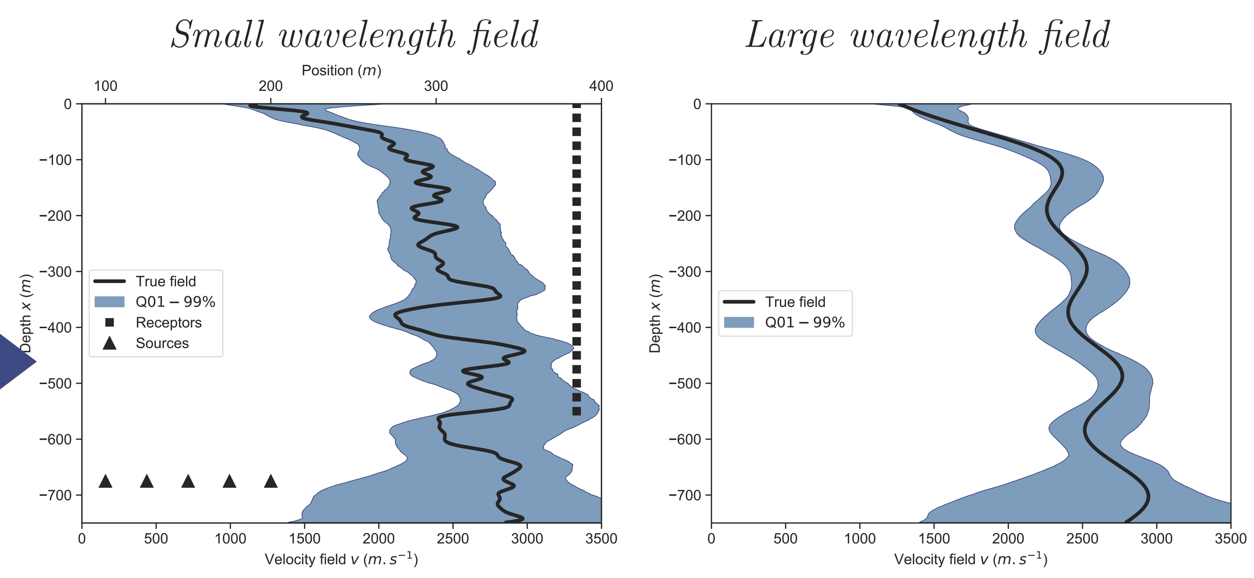
(seismic tomography problem, inspired by the realistic model Amoco Tulsa Research Lab (O'Brien, 1994))

Posterior field distributions with fixed hyperparameters value

The posterior is strongly constrained by the choice of l and using a fixed l does not allow distinguishing various field shapes.



Posterior field distributions with the change of measure method.



The sampling of the hyperparameters space improves the uncertainties estimation and allows distinguishing the two fields.

6 Conclusion and perspectives

- The CoM is a **fast** and **accurate** method to provide uncertainties on a **physical field**.
- The **sampling of the hyperparameters space** is highly valuable to **mitigate overconfidence** on the posterior field distribution.
- Further work:
  - Propagation of the uncertainties to source location
  - Adaptive inference

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